

# Isoperiodic classical systems and their quantum counterparts

M. Asorey<sup>a</sup>, J. F. Cariñena<sup>a</sup>, G. Marmo<sup>b,c</sup>, A. Perelomov<sup>d</sup>

<sup>a</sup> *Departamento de Física Teórica, Facultad de Ciencias Universidad de Zaragoza, 50009 Zaragoza, Spain*

<sup>†b</sup> *Dipartimento di Scienze Fisiche, Università Federico II di Napoli*

<sup>c</sup> *and INFN, Sezione di Napoli, Complesso Univ. di Monte Sant'Angelo, Via Cintia, 80125 Napoli, Italy*

<sup>d</sup> *Institute for Theoretical and Experimental Physics, 117259 Moscow, Russia.*

---

## Abstract

One-dimensional isoperiodic classical systems have been first analyzed by Abel. Abel's characterization can be extended for singular potentials and potentials which are not defined on the whole real line. The standard shear equivalence of isoperiodic potentials can also be extended by using reflection and inversion transformations. We provide a full characterization of isoperiodic rational potentials showing that they are connected by translations, reflections or Joukowski transformations. Upon quantization many of these isoperiodic systems fail to exhibit identical quantum energy spectra. This anomaly occurs at order  $\mathcal{O}(\hbar^2)$  because semiclassical corrections of energy levels of order  $\mathcal{O}(\hbar)$  are identical for all isoperiodic systems. We analyze families of systems where this quantum anomaly occurs and some special systems where the spectral identity is preserved by quantization. Conversely, we point out the existence of isospectral quantum systems which do not correspond to isoperiodic classical systems.

*Key words:* Isoperiodicity. Shear equivalence. Isospectral potentials. Quantum anomalies. Darboux transformation. Joukowski transformations

---

## 1 Introduction

The connection between classical and quantum physics has always been tantalizing and elusive. The establishment of quantization rules for classical system has been the algorithmic method which dominated the construction of quantum systems. This pathway has been plagued with surprises: existence of quantum anomalies, operator ordering problems, quantum divergences, spontaneous symmetry breaking, renormalization of couplings and observables, etc. The way back to classical mechanics from quantum dynamics has revealed also problematic due to the failure of semiclassical expansion and the existence of quantum states without a natural classical analogue. One of the most explicit realizations of the genuine differences between classical and quantum systems is provided by the analysis of boundary conditions in systems evolving in constrained spaces [1,2]. However, this mismatch has been very useful to introduce new quantum inspired classical structures: quantum groups, non-commutative geometry, etc.

In this note we explore the analogies and differences between the equivalences of classical and quantum systems from a spectral point of view. There is a natural equivalence relation between classical mechanical systems based on the analysis of the periods of closed orbits and its dependence on the orbit energy. Two bounded mechanical systems might be considered equivalent if they employ the same time periods for closed orbits with the same energy, and then they are said to be isoperiodic. This equivalence relation was introduced by Abel in 1826 [3]. It can be shown that the equivalence classes of equivalent potentials include potentials related by shear transformations but this does not exhaust all possibilities as we will show below. This fact is related with the concept of Steiner symmetrization which was used in [4] to establish that all potentials with only one minimum having the same Steiner symmetrized potential have the same dependence of periods on the energy  $T(E)$ .

Another open problem is the characterization of all potentials which give rise to isochronous motions, i.e. the period does not depend on the energy (see e.g. [5,6], and references therein). The origin of the problem is even older, it goes back to Huygens in 1673 [7]. In the one-dimensional case with rational potentials it can be shown that the only symmetric isochronous potentials with a constant period  $T = 2\pi/\omega$  correspond either to the harmonic oscillator  $U(x) = \frac{1}{2}\omega^2 x^2$  or to the isotonic potential of the form  $U(x) = \frac{1}{8}\omega^2 x^2 + \alpha x^{-2}$ , up to a translation [8].

There is a similar equivalence relation for quantum systems. Two quantum systems with bounded classical analogs are said to be spectrally equivalent if their energy levels are identical. It is well known that the isoperiodicity equivalence is the classical version of the quantum isospectrality condition (see e.g. [9] and [10] for a recent discussion). In the same way it is obvious that the quantum counterpart of isochronicity is the harmonic spectrum (for regular potentials). However, there is not a theorem characterizing the potentials with equally spaced energy levels in an analogous way as for the isochronous systems. In particular, we shall show the existence of many isochronous classical systems which do

not have equally spaced quantum energy spectra.

More generally, the classical equivalence associated to isoperiodicity is not always preserved by the quantization process, i.e. given two isoperiodic classical systems the corresponding quantum systems might not be isospectral. The exploration of the anomalies of this correspondence is one of the goals of this paper. In particular, we will exhibit many isochronous classical systems which do not have equally spaced quantum energy spectra. Conversely, we will also show that there are spectrally equivalent quantum systems which are not classically isoperiodic.

In the path integral approach to quantum mechanics the anomaly can be understood by the simple fact that the contribution of paths which do not correspond to classical solutions of motion equations give different contributions for some isoperiodic potentials. However, the equivalence between isoperiodic classical systems is not broken in the semiclassical approximation  $\mathcal{O}(\hbar)$ . Thus, the quantum anomalies, when they exist, can only appear in higher order corrections  $\mathcal{O}(\hbar^2)$ .

Isoperiodic deformations of an potential are of two types: shear transformations and time-space scale transformations. The difference between both deformations is that in the first case the energy of the orbits is preserved whereas in the second case the energy levels are scaled. The quantum anomaly in the first case can be interpreted as an obstruction to the shear transformation which requires an additional amount of energy to be performed unlike for the classical systems. On the contrary, the scale transformation always involves energy transfer in both cases. One of the main results of the paper is the proof that the full characterization of isoperiodic rational potentials can be achieved in terms of translations, reflections and Joukowski transformations.

On the other hand, there are quantum mechanical systems whose potentials are related by a Darboux transformation that implies that they have (almost) identical energy spectra. Some of those systems turn out to be classically isoperiodic but some others do not. These facts illuminate the relations existing between quantum isospectrality and classical isoperiodicity, two similar but not identical dynamical concepts. The analysis of these equivalences at the classical and quantum levels provides a very illuminating picture of the quantum/classical transition.

The paper is organized as follows: In Section 2 we analyze the notion of isoperiodicity and provide a characterization of polynomial isochronous potentials. The generalization of Abel's theory for singular potentials is approached in Section 3, where we prove the main results of the paper concerning the characterization of isoperiodic rational potentials which are illustrated with some illuminating examples. In Section 4 we analyze the role of scale invariance in the analysis of isoperiodicity. The quantum analogue of isoperiodicity is isospectrality. The appearance of anomalies in the quantization of isoperiodic potentials prevents the isospectrality of some isoperiodic potentials, although their first order semiclassical corrections are identical. This is shown in Section 5 while in Section 6 we analyze the opposite case, where we analyze some isospectral potentials related by

Darboux transforms which are not classically isoperiodic.

## 2 Isoperiodic Potentials

The complete identification of all potentials of one-dimensional mechanical systems giving rise to the same dependence  $T(E)$  of the period  $T$  of recurrent trajectories with an energy  $E$  was provided by Abel [3] (see also [11]). This classification can also be obtained by identifying the deformations of the potential that do not change the  $T(E)$  dependence. For one dimensional problems this period/energy dependence is given by

$$T(E) = \sqrt{2m} \int_{x_m(E)}^{x_M(E)} \frac{dx}{\sqrt{E - U(x)}}. \quad (1)$$

where  $m$  is the mass of the particle and  $x_m(E)$  and  $x_M(E)$  denote the two turning points which are the roots of the equation  $U(x) = E$ .

Here  $U(x)$  will be assumed to be a convex potential of the form displayed in Figure 1. Having in mind the invariance under translations, we can assume in the simplest case, the asymptotic behavior  $\lim_{x \rightarrow \pm\infty} U(x) = \infty$  and that  $U$  has two branches,

$$U(x) = \begin{cases} U_1(x) & \text{if } x < 0 \\ U_2(x) & \text{if } x > 0 \end{cases}$$

where  $U_1(x)$  and  $U_2(x)$  are two monotone decreasing/increasing functions, i.e. such that  $xU'(x) > 0$ . The inverse maps of  $U_1(x)$  and  $U_2(x)$  will be denoted  $x_1(U)$  and  $x_2(U)$ , respectively. Note that their values for  $U = E$  are those of the turning points.

For more general non-convex potentials like those of Figure 2 the period does not depend only on the energy  $E$  but also depends on the periodic branch specified by  $x_m(E)$  and  $x_M(E)$ , i.e.  $T(E, x_m, x_M)$ . Note that such type potentials cannot be isochronous [8].

We shall restrict ourselves in this section to the convex case and we shall postpone the discussion of other interesting cases, for instance those in which the potential presents poles, for next sections. The existence of nodes splits the one-dimensional space into isolated domains bounded by the poles where the dynamics of the system is confined.

It was shown by Abel [3] that the relation between energy and period given by (1) does not uniquely determine the potential  $U$ , but only the difference  $x_2(U) - x_1(U)$ , which is

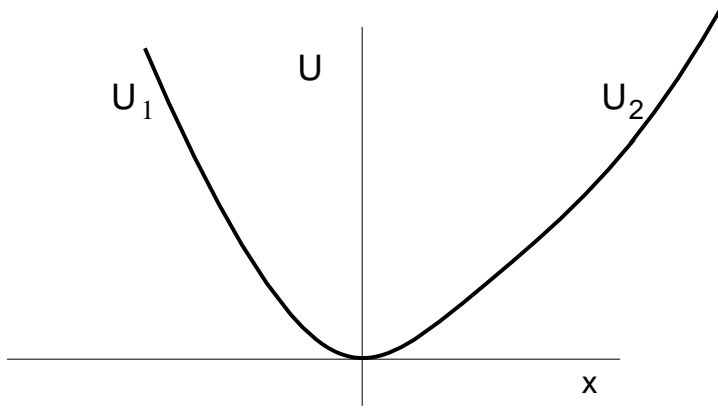


Fig. 1. Generic convex potential

given by:

$$x_2(U) - x_1(U) = \frac{1}{\pi \sqrt{2m}} \int_0^U \frac{T(E)}{\sqrt{U-E}} dE . \quad (2)$$

For a proof using Laplace transformation see e.g. [17]. This expression shows that the general potential  $U(x)$  having a given period/energy dependence can be expressed by means of a particular solution  $x_i^0$ ,  $i = 1, 2$ , and choosing an arbitrary function  $g : \mathbb{R} \rightarrow \mathbb{R}$ , and in terms of  $g$  the general solution of the Abel equation is

$$x_2(U) = x_2^0(U) + g(U), \quad x_1(U) = x_1^0(U) + g(U) . \quad (3)$$

Obviously  $g$  should not alter the assumed convex character of the potential.

In particular, if we choose  $g(U) = -\frac{1}{2}(x_1^0(U) + x_2^0(U))$  we find a solution  $(x_1^s, x_2^s)$  such that  $x_1^s(U) = -x_2^s(U)$ ,

$$x_2^s(U) = \frac{1}{2}(x_2(U) - x_1(U)), \quad x_1^s(U) = -x_2^s(U) ,$$

and therefore corresponding to a potential  $U^s$  which is symmetric under reflection with respect to the origin, i.e.  $U^s(-x) = U^s(x)$ . Such potential is nothing but the Steiner symmetrization [4] of the starting potential  $U$ . Using such a particular solution, the general solution is given by

$$x_2(U) = x_2^s(U) + g(U), \quad x_1(U) = -x_2^s(U) + g(U) . \quad (4)$$

Under the additional assumption that the potential  $U$  is symmetric under reflection with respect to the origin there is a unique potential  $U$  with a given period/energy dependence

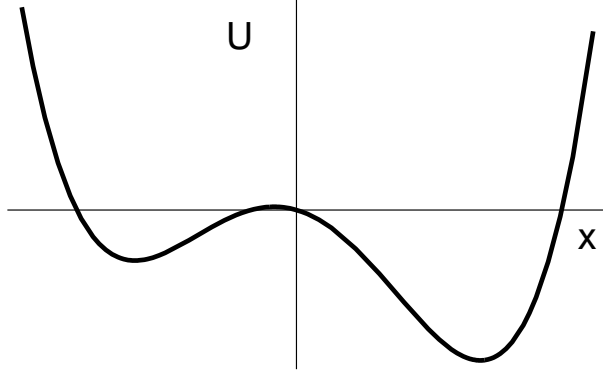


Fig. 2. Non-convex potential

satisfying (1), which will be the one given for  $U > 0$  by [12,13,14]:

$$x_2^s(U) = \frac{1}{2\pi\sqrt{2m}} \int_0^U \frac{T(E)}{\sqrt{U-E}} dE . \quad (5)$$

Now, because of the convex character of  $U$  the relation (5) can be inverted giving  $U^s$  as a function of  $x_2^s$  for  $x_2^s \geq 0$ .

Note that relation (3) can be inverted giving rise to a relation

$$U(x) = U^0(x + f(U(x))) , \quad (6)$$

and more particularly, when  $U^0$  is the symmetric potential  $U^s$  in its equivalence class of potentials, we obtain from (4) the relation

$$U(x) = U^s(x + g(U(x))) .$$

There is a very fundamental identity which characterizes all potentials (related by a shear transformation) having the  $T_U(E)$  dependency of the period as a function of the energy corresponding to the potential  $U$ :

$$U(x) = U(x + W_U(U(x))) \quad \text{where} \quad W_U(V) = \frac{1}{\pi\sqrt{2m}} \int_0^V \frac{T_U(E)}{\sqrt{V-E}} dE , \quad (7)$$

which is easily derived from Abel relation (2).

The potentials in the same class of shear equivalence as a given potential  $U$  can be characterized as the fixed points of the following transformation

$$\tilde{U}(x) = \bar{U}(x + W_U(\bar{U}(x))) . \quad (8)$$

This transformation can be considered as a classical analog of a renormalization group transformation. When compared with (6), it shows that it is completely characterized by the choice of the basic potential  $U$  which determines the nature of the fixed point potentials which are shear equivalent to  $U$ .

This renormalization group transformation is very useful to further characterize the isoperiodic potentials within a certain class of potentials. In particular, is very useful to prove some theorems concerning isochronous rational potentials.

It is commonly believed that the harmonic oscillator is the only polynomial potential which is isochronous. This guess can be substantiated in more rigorous terms [15] .

**Theorem 1.** A convex polynomial potential  $U(x)$  is isochronous iff  $U(x) = ax^2 + bx + c$

*Proof:* If the potential  $U(x)$  is an isochronous potential with period  $T$  we can use (2) when  $T(E)$  is constant and we obtain

$$x_2(U) - x_1(U) = \frac{1}{\pi\sqrt{2m}} \int_0^U \frac{T}{\sqrt{U-E}} dE = \frac{2T}{\pi\sqrt{2m}} \sqrt{U},$$

which implies that [8]

$$U(x) = U\left(x + \frac{2T}{\pi\sqrt{2m}} \sqrt{U(x)}\right). \quad (9)$$

From the analysis of the leading term of (9) we see that a solution  $U$  of (9) can be polynomial only if  $U$  is the square of a linear polynomial, i.e.  $U(x) = (\alpha x + \beta)^2$  which proves the theorem. The condition  $U(0) = 0$  fixes  $\beta = 0$  and yields to the standard harmonic oscillator.

A generalization for the case of rational functions is also possible and will be considered in next section.

Let us examine some examples of the isochronous case for which  $T(E)$  takes a constant value  $T$ . In that case, using (5) we see that  $x^s(U) = (T/\pi)\sqrt{U/(2m)}$  and the general solutions  $x_1(U)$  and  $x_2(U)$  are, respectively [3],

$$x_1(U) = -\frac{T}{\pi}\sqrt{\frac{U}{2m}} + g(U), \quad x_2(U) = \frac{T}{\pi}\sqrt{\frac{U}{2m}} + g(U). \quad (10)$$

**Case A.** If we choose  $g(U) = a$  we find

$$x_1(U) = -\frac{T}{\pi}\sqrt{\frac{U}{2m}} + a, \quad x_2(U) = \frac{T}{\pi}\sqrt{\frac{U}{2m}} + a, \quad (11)$$

from which we obtain the harmonic oscillator potential centered at  $x = a$ .

$$U(x) = \frac{m\omega^2}{2}(x-a)^2, \quad \text{with } \omega = \frac{2\pi}{T}. \quad (12)$$

**Case B.** If, instead, the function  $g$  is chosen as  $g(U) = \alpha(T/\pi)\sqrt{U/(2m)}$ , then

$$x_1(U) = (-1 + \alpha) \frac{T}{\pi} \sqrt{\frac{U}{2m}}, \quad x_2(U) = (1 + \alpha) \frac{T}{\pi} \sqrt{\frac{U}{2m}}, \quad (13)$$

which for  $|\alpha| \neq 1$ , corresponds to the potential of two half-oscillators [17,18,19] sometimes called split-harmonic oscillator [16,10]:

$$U(x) = \begin{cases} \frac{1}{2} m \omega_1^2 x^2 & \text{if } x \leq 0 \\ \frac{1}{2} m \omega_2^2 x^2 & \text{if } x \geq 0 \end{cases} \quad (14)$$

with different angular frequencies

$$\omega_1 = \frac{2\pi}{(1-\alpha)T} = \frac{\omega_0}{1-\alpha}, \quad \omega_2 = \frac{2\pi}{(1+\alpha)T} = \frac{\omega_0}{1+\alpha}, \quad (15)$$

glued together at the origin of coordinates [18]. Note that

$$1/\omega_1 + 1/\omega_2 = 2/\omega_0, \quad (16)$$

where  $\omega_0 = 2\pi/T$ . The harmonic oscillator with  $\omega = \omega_0/2$  is the Steiner symmetrized of this split-harmonic oscillator [4].

Conversely, given a potential like in (14) we can reduce it to Case B with the choice [17]

$$\alpha = \frac{\omega_1 - \omega_2}{\omega_1 + \omega_2} \quad \text{and} \quad \omega_0 = \frac{2\omega_1\omega_2}{(\omega_1 + \omega_2)}.$$

Note that this potential (14) is not analytic at  $x = 0$  but  $U''(0+) - U''(0-) = m(\omega_2^2 - \omega_1^2)$ .

The limit cases  $\alpha = \pm 1$  correspond to the half harmonic oscillator and its reflected one, to be studied later.

### 3 Singular potentials and shear equivalence

There are two slight generalizations of Abel theorem for non-convex and singular potentials. The first one arises as a consequence of the Euclidean symmetry of the kinetic



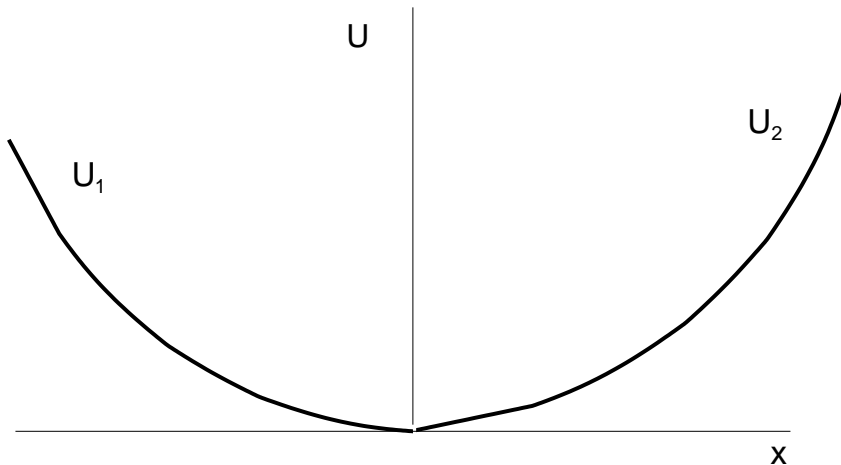


Fig. 3. Split-harmonic oscillator with two different frequencies  $\omega_1, \omega_2$

term of mechanical systems. In particular, it is invariant under space translations and reflections. The space translation symmetry is the cause for the ambiguity in isoperiodic systems associated to the choice of the shear function  $g(U) = a$ . Now, the reflection symmetry interchanges the order of turning points of closed trajectories and establishes the mechanical equivalence (isoperiodicity) of a potential  $U$  and its space reflected pair  $U^R(x) = U(-x)$ , for which  $x_1^R(U) = -x_2^R(U)$  and  $x_2^R(U) = -x_1^R(U)$ . It is obvious that this operation preserves the relations period/energy for any potential. In the case of convex potentials the equivalence is included in the Abel family of isoperiodic solutions. However, for non-convex potentials the reflection transformation introduces a new type of solution not included in Abel's family of isoperiodic potentials. The most general solution for any smooth potential is thus given from a particular solution  $U_*$ , its reflected pair  $U_*^R$  and their shear equivalents

$$U_g(x) = U_*(x - g(U_g(x))) \quad U_g^R(x) = U_*(-x + g(U_g(-x))). \quad (17)$$

The reflection symmetry could be in principle defined with respect to any point of the real line, but the isoperiodic potentials obtained by this transformation are included in those of (17) because the most general reflection can be expressed as a composition of reflections with respect to the origin and translations, both considered in (17).

The second generalization of Abel's solution (3) concerns the case of singular potentials or potentials which are not defined on the whole real line. In that case one has to look for new types of isoperiodic potentials. Let us analyze once again the isochronous case. In that case we have a generalization of Theorem 1 for the case of rational potentials.

**Theorem 2.** A rational potential  $U(x)$  which does not reduce to a polynomial is isochronous iff <sup>1</sup>

---

<sup>1</sup> This theorem was first proved by Chalykh and Veselov [8]. The proof below is a different proof.

$$U(x) = \left( \frac{ax^2 + bx + c}{x + d} \right)^2.$$

*Proof:* Any rational potential  $U(x)$  solution to (9) requires that  $U(x)$  has to be the square of the irreducible quotient of two polynomials  $P(x)$  and  $Q(x)$ ,

$$U(x) = \left( \frac{P(x)}{Q(x)} \right)^2.$$

The stability of the leading term under the non-linear constraint (9) requires that the degree of  $P$  cannot be higher than one unit more than that of  $Q$ . As  $U(x)$  was assumed to be rational we can consider the analytic continuation of such a function to the complex plane. If  $Q$  is not constant the potential develops at least one pole in the complex plane which is not a zero of  $P$  because  $P$  and  $Q$  cannot have common zeros. We will show that  $Q(x)$  cannot have two different zeros and therefore that in such a case the zero should be real. Indeed, if  $w$  is a zero of  $Q$  let us consider the function  $R_w(z)$  given by

$$R_w(z) = Q(z)(z - w) - \frac{2T}{\pi\sqrt{2m}}P(z). \quad (18)$$

Such a function cannot have a zero, because if we assume that  $R_w(\zeta) = 0$ , and  $Q(\zeta) \neq 0$ , then  $\zeta$  is the partner of  $w$  because

$$\zeta = w + \frac{2T}{\pi\sqrt{2m}} \frac{P(\zeta)}{Q(\zeta)},$$

and therefore  $\zeta$  is a pole of  $U(z)$ , what is not possible because we assumed that  $Q(\zeta) \neq 0$ . On the other side, had we assumed that  $\zeta$  is a zero of  $R_w$  for which  $Q(\zeta) = 0$ , then (18) shows that also  $P(\zeta) = 0$ , what is once again against our hypothesis that  $P$  and  $Q$  have no common zeroes.

As the polynomial function  $R_w(z)$  has not zeroes, it should be a constant. Now, if we have two different zeros of  $Q$ ,  $w_1$  and  $w_2$ , the preceding argument shows the existence of two constants  $c_1$  and  $c_2$  such that

$$Q(z)(z - w_1) - \frac{2T}{\pi\sqrt{2m}}P(z) = c_1, \quad Q(z)(z - w_2) - \frac{2T}{\pi\sqrt{2m}}P(z) = c_2,$$

from where we find that

$$w_1 - w_2 = \frac{c_2 - c_1}{Q(x)},$$

which implies that  $Q$  must be a constant, reducing the problem to the previously considered case of  $U$  being polynomial. Therefore the only possible non-polynomial solution is the one given by a polynomial  $P$  of degree two and a polynomial  $Q$  of degree one with one single real zero, which completes the proof of the claim.

Note that using translational symmetry we can fix the real pole at  $x = 0$ , (i.e.  $d = 0$ ) and the classical motion can be then restricted to the open interval  $(0, \infty)$ .

Some other examples with a non-analytic behavior are the following.

**Case C.** The half harmonic oscillator whose potential is

$$U(x) = \begin{cases} \infty & \text{if } x \leq 0 \\ \frac{1}{2}m\omega^2 x^2 & \text{if } x \geq 0 \end{cases} \quad (19)$$

is only defined in half a line. However it does not define a new family of isochronous potentials because it can be included in the Abel's family of the regular harmonic oscillator  $U(x) = 2m\omega^2 x^2$ . In fact, it is related to the oscillator by the shear transformation defined by [17]

$$g(U) = -\frac{\sqrt{U}}{\sqrt{2m\omega^2}}. \quad (20)$$

Note that this half-harmonic system can be considered as the limit when  $\omega_1$  tends to infinity of the two half-oscillators system (14). In fact, using the relation (16) with  $\omega_2 = \omega$  and taking the limit when  $\omega_1$  tends to  $\infty$  we obtain  $\omega_0 = 2\omega$  and therefore the potential (19) is in the same equivalence class as the harmonic oscillator given by  $U(x) = 2m\omega^2 x^2$ .

Finally, as indicated before, this potential is obtained in Case B for  $\alpha = 1$ .

**Case D.** To the same family belongs the potential [8,20,21,22]

$$U(x) = \frac{2\alpha^2}{m\omega^2 x^2} + \frac{1}{2}m\omega^2 x^2 - 2\alpha = \frac{1}{2}m\omega^2 \left( \frac{2\alpha}{m\omega^2 x} - x \right)^2. \quad (21)$$

It is obvious that this potential is isochronous because in fact it is related with the half harmonic oscillator (19), by means of a shear transformation

$$g(U) = \sqrt{\frac{U}{2m\omega^2}} - \sqrt{\frac{4\alpha + U}{2m\omega^2}} \quad (22)$$

and to the symmetric oscillator by the shear transformation [17]

$$g(U) = -\sqrt{\frac{4\alpha + U}{2m\omega^2}} \quad (23)$$

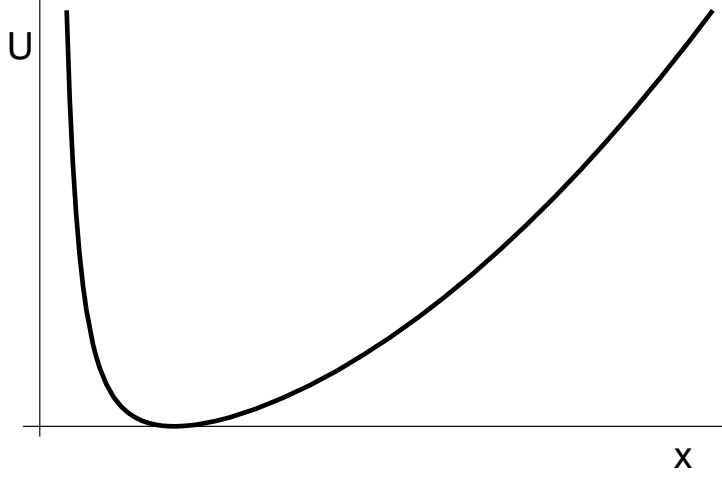


Fig. 4. Isochronous potential  $U(x) = \frac{1}{2}m\omega^2 \left(\frac{2\alpha}{m\omega^2 x} - x\right)^2$

Note that according to Chalykh-Vesselov theorem [8] (theorem 2), this potential and the harmonic oscillator potential are the only rational isochronous potentials.

Another characteristic case of the same family is the following isochronous one:

**Case E.** For  $g(U) = \alpha U$ , then

$$x_1(U) = -\sqrt{\frac{2U}{m\omega^2}} + \frac{2\alpha U}{m\omega^2}, \quad x_2(U) = \sqrt{\frac{2U}{m\omega^2}} + \frac{2\alpha U}{m\omega^2}, \quad (24)$$

and therefore, for both values of  $x$  we have

$$\left(x - \frac{2\alpha U}{m\omega^2}\right)^2 = \frac{2U}{m\omega^2} \quad (25)$$

or in other form,

$$\alpha^2 \frac{U^2}{m^2\omega^4} - \left(\alpha x + \frac{1}{2}\right) \frac{U}{m\omega^2} + \frac{1}{4}x^2 = 0, \quad (26)$$

from which we obtain that [18], if  $x \geq -1/(4\alpha)$ ,

$$U(x) = \frac{m\omega^2}{2} \left[ \frac{x}{\alpha} + \frac{1}{2\alpha^2} - \frac{1}{\alpha} \sqrt{\frac{x}{\alpha} + \frac{1}{4\alpha^2}} \right]. \quad (27)$$

Note that  $U(0) = 0$  and for small values of  $x$ ,

$$U(x) \approx \frac{1}{2}m\omega^2 x^2 - m\alpha\omega^2 x^3 + \dots \quad (28)$$

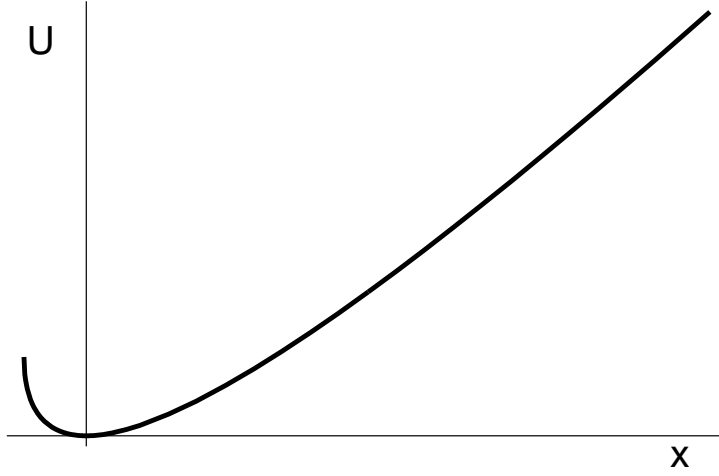


Fig. 5. Isochronous potential  $U(x) = \frac{m\omega^2}{2} \left[ \frac{x}{\alpha} + \frac{1}{2\alpha^2} - \frac{1}{\alpha} \sqrt{\frac{x}{\alpha} + \frac{1}{4\alpha^2}} \right]$

In this case, although the shear transformation of the oscillator is smooth the final system is only defined on half a line.

**Case F.** An archetypal case is the reduced Kepler problem (see [11], Chapter III)

$$U(x) = -\frac{e^2}{x} + \frac{l^2}{2mx^2} \quad \text{for } x > 0 \quad (29)$$

whose period function for negative energies, is well known

$$T(E) = \pi e^2 \sqrt{\frac{m}{2|E|^3}} \quad (30)$$

and is a particular case of a more general family of potentials (see also [11], Chapter III)

$$U(x) = A |x|^n \quad (31)$$

with periods

$$T(E) = \frac{2}{n} \sqrt{\frac{2\pi m}{E}} \left( \frac{E}{A} \right)^{\frac{1}{n}} \frac{\Gamma\left(\frac{1}{n}\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{n}\right)}. \quad (32)$$

**Case G.** A very peculiar different example is the infinite wall,

$$U(x) = \begin{cases} 0 & \text{if } x \in [0, \pi] \\ \infty & \text{if } x \notin [0, \pi] \end{cases} \quad (33)$$

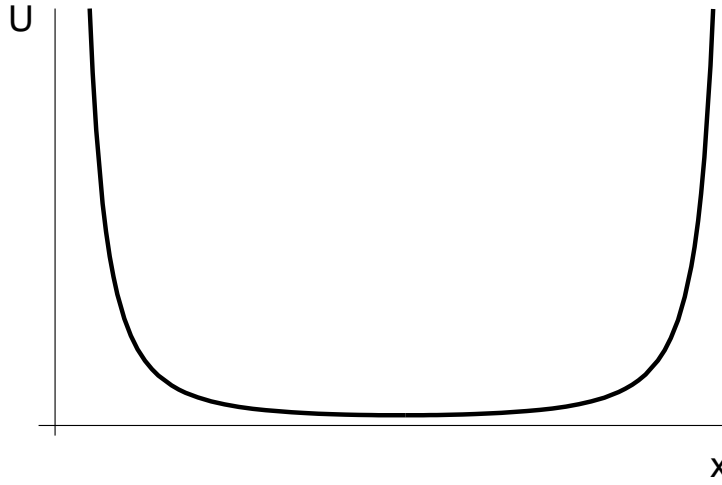


Fig. 6. Smooth well potential  $U(x) = \frac{1}{\sin^2(x)}$  with the same energy spectrum that the infinity square well.

which is only shear equivalent to itself up to space translations. In this case, the Abel inverse of the period function

$$T(E) = \pi \sqrt{\frac{2m}{E}} \quad (34)$$

is uniquely defined up to a shift by a real constant  $a$ .

**Case H.** A similar potential with the same quantum energy spectrum

$$U(x) = \frac{1}{m} \left( \frac{1}{\sin^2(x)} - \frac{1}{2} \right) \quad (35)$$

has a much larger degeneracy [23].

The last two cases show that the orbits of Abel's shear transformations are not of the same type.

For even potentials there is a special case of shear transformation which preserves the periods. It is given by a composition of an inversion with two translation transformations.

The transformation of the complex plane called Joukowski transformation, defined by  $J_\lambda(z) = z + (\lambda/z)$ , with  $\lambda \in \mathbb{R}$ , plays a relevant in aerodynamics applications. We consider here an analogous map of the real line completed with the two points at the infinity:

$$J_g(x) = \frac{x}{2} - \frac{2g^2}{x}, \quad .$$

We also consider the involution of  $\bar{\mathbb{R}}$ ,  $i_g : \bar{\mathbb{R}} \rightarrow \bar{\mathbb{R}}$ , given by

$$i_g(x) = -\frac{4g^2}{x}. \quad (36)$$

Note that  $J_g(0+) = -\infty$  and  $J_g(\pm\infty) = \pm\infty$  and the important property  $J_g \circ i_g = J_g$ . Consequently, the points  $x$  and  $-4g^2/x$  have the same image. Moreover, only these two points have the same image, because if  $x/2 - 2g^2/x = y$ , then  $x^2 - 2xy - 4g^2 = 0$ , and therefore the two roots are given by

$$x_{\pm}(y) = y \pm \sqrt{y^2 + 4g^2},$$

i.e.  $x_+(y) > 0$ ,  $x_-(y) < 0$  and  $x_+(y)x_-(y) = -4g^2$ .

We can use the properties of these transformations  $J_g$  and  $i_g$  to prove:

**Theorem 3.** If  $U(x)$  is a bounded below even convex potential with  $\lim_{x \rightarrow \infty} U(x) = \infty$ , then for any real number  $g$  the potential  $U_g$  given by

$$U_g(x) = U(J_g(x)) = U\left(\frac{x}{2} - \frac{2g^2}{x}\right)$$

is isoperiodic with  $U(x)$ .

*Proof:* First notice that  $U_g$  is invariant under the transformation  $i_g$ , because  $U_g(i_g(x)) = U(J_g(i_g(x))) = U(J_g(x)) = U_g(x)$ . The parity symmetry of the function  $U$  implies that  $U_g(x) = U_g(4g^2/x)$ .

On the other side, as the function  $U$  is a bounded below even convex potential the minimum of the potential is at the origin and we can consider without any restriction that the minimum value of is  $U(0) = 0$ .

If  $U_g(x_1) = U_g(x_2)$ , then  $U(J_g(x_1)) = U(J_g(x_2))$ , and therefore, given an arbitrary positive energy value  $E > 0$  there will be two real numbers  $x_-(E) < 0$  and  $x_+(E) > 0$  such that  $-x_-(E) = x_+(E)$  and  $U(x_{\pm}(E)) = E$ . Consequently, using the definition of the new potential function  $U_g$ , there will exist four points, to be denoted  $x_{g_1}^-, x_{g_2}^-, x_{g_1}^+, x_{g_2}^+$  such that  $U_g(x_{g_i}^{\pm}) = E$ . They are respectively given by

$$\begin{aligned} x_{g_1}^- &= -x_+(E) - \sqrt{(x_+(E))^2 + g^2}, & x_{g_2}^- &= x_+(E) - \sqrt{(x_+(E))^2 + g^2}, \\ x_{g_1}^+ &= -x_+(E) + \sqrt{(x_+(E))^2 + g^2}, & x_{g_2}^+ &= x_+(E) + \sqrt{(x_+(E))^2 + g^2}. \end{aligned}$$

The span between the two  $U_g$ -equipotential values  $x_{g_1}^-$  and  $x_{g_2}^-$ , and same for  $x_{g_1}^+$  and  $x_{g_2}^+$ , is  $2x_+(E)$ , and it coincides with the span between the corresponding  $U$ -equipotential

values of the parity symmetric potential  $U$ . Consequently, the potentials  $U_g$  and  $U$  are shear related and isoperiodic<sup>2</sup>.

In the case of isochronous potentials this connection between pairs of potentials provides us with the only solutions to isochronous rational potentials in terms of the harmonic oscillator and the isotonic potential (see theorem 2). This result can be further generalized. Indeed it can be shown that in the rational case these two families of potentials are the only ones which are isoperiodic not only for the isochronous periods but for any frequency-energy spectral distribution associated to a rational potential.

**Theorem 4.** Any non-trivial rational potential  $U_*$  which is isoperiodic to a given even convex polynomial potential  $U$  is either of the form  $U_c = U(x + c)$  or  $U_g^c(x) = U((x - c)/2 - 2g^2/(x - c))$ , for any value of  $g$ .

*Proof:* If there is a rational potential solution of

$$U_*(x) = U_*(x + W_U(U_*(x))). \quad (37)$$

the function  $W_U(U_*(x))$  has to be the irreducible ratio of two polynomials  $P(x)$  and  $Q(x)$ , i.e.

$$W_U(U_*(x)) = \frac{P(x)}{Q(x)}.$$

The stability of the leading term under the non-linear constraint (9) requires that the degree of  $P$  has to be one unit larger than that of  $Q$ . On the other hand by construction the transformation  $K$  defined by

$$K(x) = x + \frac{P(x)}{Q(x)}$$

has to be invertible and involutive, i.e.  $K \circ K = \text{Id}$ . It is easy to show that the only rational solutions satisfying this requirement are

$$K_c(x) = -x + c \quad K_c^g(x) = c - \frac{4g^2}{x - c}, \quad (38)$$

which correspond to the kind of transformations, *translations, reflections and inversions*, described previously in this section.

Indeed, if  $K_c$  is polynomial the asymptotic analysis at  $x \sim \infty$  requires that the leading term  $K_c \sim a_n x^n$  satisfies  $K_c(K_c(x)) \sim a_n^{n+1} x^{n^2} \simeq x$  and thus  $n = 1$  and  $a_n^2 = 1$ . The

---

<sup>2</sup> Strictly speaking  $U_g$  has two branches, one in each half-line of positive/negative values of  $x \in \mathbb{R}$ . Therefore, there is a degeneracy of trajectories which is not present in the convex potential  $U$  which has only one branch.



only non-trivial solutions of these requirements are the translations/reflections of (38). This regular type of solutions keeps the polynomial character of the potential and simply involve a reflection and a translation of the polynomial.

If  $K_c$  is rational it can have poles in the complex plane. Notice that because of the rational character of the transformation  $K_c$  the involutive property can be extended to the whole complex plane. If  $K_c$  is not a pure polynomial it has to have a pole at a point  $c \neq \infty$  which is the image of  $x = \infty$ . and can be rewritten in the form

$$K_c(z) = c - \frac{P_0(z)}{(z - c)Q_0(z)}. \quad (39)$$

Since

$$K_c \circ K_c(z) = c - \frac{P_0(K_c(z))Q_0(z)(z - c)}{P_0(z)Q_0(K_c(z))} = z \quad (40)$$

we have that

$$1 = \frac{P_0(K_c(z))Q_0(z)}{Q_0(K_c(z))P_0(z)}. \quad (41)$$

It is easy to show that the only solution is  $Q_0(z) = P_0(z) = \text{cte}$  and the pole has to be a real pole, i.e  $c^* = c$ . The absence of other poles is excluded by the involutive character of the transformation, i.e. only one point in the complex plane can be involutively mapped into  $z = \infty$ .

The second kind of solutions of (38) is more subtle and implies that  $P = 4g^2 - (x - c)^2$  and  $Q = x - c$ , which means that  $W_U(U_*(x))$  and therefore  $U_*$  develops a single pole singularity at  $x = c$ . In this case  $U_*$  is symmetric under inversion transformations,

$$U_*(x) = U_* \left( c + \frac{4g^2}{x - c} \right) \quad (42)$$

the same symmetry properties that the potential  $U_g^c$  satisfies. Now, since  $U$  is convex even potential its minimum is attained at  $x = 0$ . The minimum of  $U_*$  for  $x > c$  is at  $x = c + 2g$  and since  $U_*$  is isoperiodic to  $U$  the values of the two potential at both minima have to be identical, i.e.  $U_*(2g) = U(0)$ . By theorem 3 the potential  $U_g^c$  is also isoperiodic to  $U$ , has the same symmetry that  $U_*$  under inversion transformations (42) and verifies that  $U_g^c(2g) = U(0)$ .

Now, since  $U_g^c$  and  $U_*$  are isoperiodic both must attain the same values at  $x$  and  $c + 4g^2/x - c$ , which implies that  $U_* = U_g^c$  and proves the theorem.  $\square$

In particular, the only singular rational potentials which are isoperiodic to  $U(x) = x^2$  and are singular at  $x = 0$  are those of the form  $U(x) = (x/2 - 2g^2/x)^2$

## 4 Scale transformations and Isoperiodicity

There is another kind of transformations which also preserves isoperiodicity. They are connected with space-time scale transformations.

The time-evolution of one-dimensional systems is described in terms of the potential function  $U(x)$ , i.e.  $\ddot{x} = -\partial U/\partial x$ . If we introduce a change of space-time coordinates defined by

$$x = \beta \tilde{x}, \quad t = \sqrt{\gamma} \tilde{t}, \quad (43)$$

where  $\beta$  and  $\gamma$  are positive real numbers, then the equation of motion becomes

$$\frac{\beta}{\gamma} \frac{d^2 \tilde{x}}{d\tilde{t}^2} = -\frac{1}{\beta} \left( \frac{\partial U}{\partial \tilde{x}} \right) (\beta \tilde{x})$$

and therefore, if we define

$$\tilde{U}(\tilde{x}) = \left( \frac{\gamma}{\beta^2} \right) U(\beta \tilde{x}), \quad (44)$$

we find that the equation of motion reads

$$\frac{d^2 \tilde{x}}{d\tilde{t}^2} = -\frac{\partial \tilde{U}(\tilde{x})}{\partial \tilde{x}}.$$

This invariance of the equation of motion is a consequence of the transformation of the action:

$$S(x) = \int dt \left( \frac{1}{2} \dot{x}^2 - U(x) \right), \quad \tilde{S} = \frac{\gamma}{\beta^2} S.$$

This suggests to study the relation between systems described by potentials  $U$  and  $\tilde{U}$  related as in equation (44).

We introduce next a generalization of a property studied by Dorignac [10].

Let  $\varphi(\zeta)$  an arbitrary function and define

$$I_\varphi(E) = \int_{x_-(E)}^{x_+(E)} \varphi(E - U(x)) dx.$$

If for any pair of real numbers  $\beta, \gamma \in \mathbb{R}$  we define a new potential given by

$$\tilde{U}(x) = \left( \frac{\gamma}{\beta^2} \right) U(\beta x),$$

then

$$\tilde{x}_{\pm}(E) = \frac{1}{\beta} x_{\pm} \left( \frac{\beta^2 E}{\gamma^2} \right),$$

and consequently

$$\tilde{I}_{\varphi}(E) = \int_{\tilde{x}_{-}(E)}^{\tilde{x}_{+}(E)} \varphi(E - U(x)) dx.$$

Therefore,

$$\tilde{I}_{\varphi}(E) = \int_{(1/\beta)x - (\beta^2 E/\gamma^2)}^{(1/\beta)x + (\beta^2 E/\gamma^2)} \varphi \left( E - \frac{\gamma^2}{\beta^2} U(\beta x) \right) dx = \frac{1}{\beta} \int_{x - (\beta^2 E/\gamma^2)}^{x + (\beta^2 E/\gamma^2)} \varphi \left( E - \frac{\gamma^2}{\beta^2} U(y) \right) dy,$$

and defining

$$\tilde{E} = \frac{\beta^2 E}{\gamma^2}$$

the equation can be rewritten as

$$I_{\varphi}(E) = \frac{1}{\beta} \int_{x_{-}(\tilde{E})}^{x_{+}(\tilde{E})} \varphi((\gamma^2/\beta^2)(\tilde{E} - U(y))) dy.$$

If the function  $\varphi$  is homogeneous of degree  $p$ ,

$$\tilde{I}_{\varphi}(E) = \frac{\gamma^{2p}}{\beta^{2p+1}} I_{\varphi}(\tilde{E}).$$

When computing the period of an oscillating motion we find a function as  $I_{\varphi}$  with  $\varphi$  a function proportional to  $\varphi_P(\zeta) = \zeta^{-1/2}$  and when computing the action we arrive to a function  $\varphi_a(\zeta) = \zeta^{1/2}$ . Therefore,

$$\tilde{I}_{\varphi_P}(E) = \frac{1}{\gamma} I_{\varphi_P}(\tilde{E}), \quad \tilde{I}_{\varphi_a}(E) = \frac{\gamma}{\beta^2} I_{\varphi_a}(\tilde{E}).$$

As a consequence If  $U(x)$  is an isochronous potential with period  $P$ , then  $\tilde{U}$  is isochronous too and its period is  $\tilde{P} = P/\gamma$ . In particular, for  $\gamma = 1$  we obtain that if  $U(x)$  is an isochronous potential with period  $P$ , then  $\tilde{U}(x) = \beta^{-2} U(\beta x)$  is isochronous too with the same frequency [10].

In particular, for  $\gamma = 1$  we see that if  $U(x)$  is an isochronous potential, then  $\tilde{U}(x) = \beta^{-2} U(\beta x)$  is isochronous too and with the same period. On the contrary, the action for this potential  $\tilde{U}$  is not the same as for  $U$ , and if the spectrum of the first one is equispaced, it will not be true for the new potential.

In many cases the above scale transformation can be shown to be equivalent to a shear transformation, e.g. in some of the examples of the previous section are related by scale transformations. However, the scale transformation has a different nature. It establishes an equivalence relation among isoperiodic potentials but does not preserve the energy levels unlike the shear transformation. For such a reason one does not expect that quantization will preserve the equivalence at the spectral level.

In physical terms the scale transformation has a energy cost whereas the classical shear transformation is energy preserving. An interesting question is to know whether or not the quantization prescription preserves this classical property. This will be the subject of next section.

## 5 Isoperiodicity and the quantum isospectrality

It is clear from the analysis of Section 2 that two potentials  $U_1$  and  $U_2$  related by a shear transformation  $g : \mathbb{R} \rightarrow \mathbb{R}^+$  define similar period functions for periodic orbits. In fact, not only the periods given by (1) are identical for  $U_1$  and  $U_2$  but also any integral between the same limits of the form

$$T_f(E) = \int_{x_m(E)}^{x_M(E)} f(E - U(x)) dx \quad (45)$$

is the same for both potentials. The proof is simple because the integral (45) can be splitted as

$$T_f(E) = \sum_{i=1}^N T_f^i(E)$$

in terms of the integrals

$$T_f^i(E) = \int_{x_i(E)}^{x_{i+1}(E)} f(E - U(x)) dx$$

where  $x_i(E)$ ,  $i = 0, 1, \dots, N$ , is the monotone sequence of points whose initial and final points are  $x_0 = x_m$ ,  $x_N = x_M$ , i.e. they coincide with the turning points of the classical trajectory, and the remaining points  $x_i$ , for  $i = 1, 2, \dots, N - 1$ , are defined by the values  $x_i \in [x_m, x_M]$  for which there is a stationary point  $x_i^* \in [x_m, x_M]$  of the potential  $U'(x_i^*) = 0$  with the same potential level  $U(x_i^*) = U(x_i)$ . In each interval  $[x_i, x_{i+1}]$  the potential function  $U(x)$  is invertible and the inverse function  $x_i(U)$  is uniquely defined. Thus,

$$T_f^i(E) = \int_{U(x_i)}^{U(x_{i+1})} dU f(E - U) x_i'(U) .$$

Now, by construction, for each interval  $[x_i, x_{i+1}]$  there is another one  $[x_{i'}, x_{i'+1}]$  such that the sum of the contributions to the the integral (45)

$$T_f^i(E) + T_f^{i'}(E) = \int_{U(x_i)}^{U(x_{i+1})} dU f(E - U) |x_i'(U) - x_{i'}'(U)| \quad (46)$$

becomes the same for the two potentials  $U_1$  and  $U_2$ . In fact, the integrand and the integral limits in (46) are identical for any couple  $U_1$  and  $U_2$  of shear equivalent potentials. In particular, this shows that  $T_f$  is also the same for all potentials related by a shear transformation and that their first semiclassical quantum corrections to the energy levels, which is given by  $T_f(E)$  with  $f(x) = \sqrt{x}$ , is also the same.

From the discussion of the previous section it follows that scale transformations (43) also preserve semiclassical corrections to energy levels if  $p = 1/2$  and  $\gamma = \beta$ . However in such a case the classical system is not isochronous.

However the higher order corrections might break the equivalence at the quantum level. The case B considered in the previous section is the simplest counterexample. The energy levels  $E_n^B$  [16] differ from those of the isoperiodic harmonic oscillator

$$E_n^A = \hbar\omega \left( n + \frac{1}{2} \right) \quad (47)$$

by terms which start at first order in perturbation theory for small values of the anharmonicity parameter  $\alpha \ll 1$

$$\begin{aligned}
E_n^B - E_n^A &= \frac{\hbar \omega_0 \alpha^2 (3 - \alpha^2)}{4(1 - \alpha^2)^2} \left( n + \frac{1}{2} \right) + \frac{\hbar \omega_0 \alpha^2}{(1 - \alpha^2)^2} \frac{2n + 1}{8} \left( (2n + 1) \psi \left( -(-1)^n \frac{n}{2} + \frac{1}{2} \right) \right. \\
&\quad \left. - (1 + 2n) \psi \left( (-1)^n \frac{1 + n}{2} + \frac{1}{2} \right) - 1 \right) + O \left( \frac{\alpha^4}{(1 - \alpha^2)^4} \right) \\
&= \hbar \omega_0 \alpha^2 \frac{2n + 1}{8} \left( 2 - (1 + 2n) \psi \left( (-1)^n \frac{1 + n}{2} + \frac{1}{2} \right) \right. \\
&\quad \left. + (2n + 1) \psi \left( -(-1)^n \frac{n}{2} + \frac{1}{2} \right) \right) + O(\alpha^4)
\end{aligned}$$

where  $\psi(x)$  is the logarithmic derivative of the Euler gamma function  $\Gamma(x)$ . The above perturbative expression for  $E_n^B$  agrees to order  $\alpha^2$  with the asymptotic behavior derived from the exact spectral equation [16]

$$\frac{\Gamma \left( \frac{3}{4} - \frac{1}{2}(1 + \alpha)E_n^B \right)}{\Gamma \left( \frac{3}{4} - \frac{1}{2}(1 - \alpha)E_n^B \right)} + \frac{\sqrt{1 + \alpha} \Gamma \left( \frac{1}{4} - \frac{1}{2}(1 + \alpha)E_n^B \right)}{\sqrt{1 - \alpha} \Gamma \left( \frac{1}{4} - \frac{1}{2}(1 - \alpha)E_n^B \right)} = 0. \quad (48)$$

One particular case where the quantum energy levels remain equal is when the shearing function  $g$  is constant,  $g = \text{const.} = g_0$ . In such a case both potentials are related by a simple translation  $U_2(x) = U_1(x - g_0)$ , which obviously does not change the quantum spectrum of the Hamiltonian. Further non-trivial examples can be obtained by means of Darboux method.

## 6 Shear deformation and Darboux transform

The quantum spectrum is also the same, up to a shift, for two potentials related by a shear transformation when they can be written in the form

$$U_1(x) = \frac{\hbar^2}{2m} \left( W(x)^2 - W'(x) \right) - a_1; \quad U_2(x) = \frac{\hbar^2}{2m} \left( W(x)^2 + W'(x) \right) - a_2, \quad (49)$$

in terms of a common superpotential  $W(x)$  with  $\lim_{x \rightarrow \pm\infty} W(x) = +\infty$  and two constants  $a_1$  and  $a_2$ . Potentials of such a type are not only related by a classical shear transformation but they are also related by a quantum Darboux transformation [24] which guarantees that the corresponding spectra of the Hamiltonians

$$H_i = \frac{p^2}{2m} + U_i \quad i = 1, 2$$

are almost identical<sup>3</sup>.

The case D of Section 3 is also an example of such a type. In fact, choosing

$$W(x) = \frac{1}{x} + x \quad (50)$$

and  $a_1 = 1 + 2\sqrt{2}$ ,  $a_2 = 3$ ,  $\hbar^2 = 2m$  we have the two potentials of case D

$$U_1(x) = \frac{2}{x^2} + x^2 - 2\sqrt{2}; \quad U_2(x) = x^2 \quad (51)$$

provided we fix  $m\omega^2 = 2$  and  $\alpha = \sqrt{2}$  for simplicity. It is also clear from the discussion of previous section that both potentials are related by the shear transformation

$$g(U) = +\frac{\sqrt{U}}{2} + \sqrt{\frac{4\sqrt{2} + U}{2}}. \quad (52)$$

More generally, for any choice of the superpotential  $W$  with parity invariance  $W(x) = W(-x)$ , i.e.  $W$  is of the form  $W(x) = K(x^2)$ , it can be shown that the corresponding potentials  $U_1, U_2$  are related by a parity symmetry  $U_1(x) = U_2(-x)$ . If  $U_1$  and  $U_2$  are convex functions they are obviously related by the shear transformation.

However, it should also be emphasized that not any pair of potentials of the form (49) related by a Darboux transform are necessarily related by a classical shear transformation. A simple counterexample is given by  $W(x) = x^4 - x$ . In that case one gets the potentials

$$U_1(x) = x^8 - 2x^5 - 4x^3 + x^2 + 1 \quad (53)$$

and

$$U_2(x) = x^8 - 2x^5 + 4x^3 + x^2 - 1, \quad (54)$$

respectively. It is obvious from the Figure 7 that both potentials are not shear related.

This illustrates that the generalization of theorems 1 and 2 to the quantum case is more sophisticated.

There are further examples of quantum isospectral systems which are not classically isoperiodic. A very interesting case is the following [25,26,27,28,29]. Let us consider a standard quantum oscillator ( $m = \omega = 1$ ) with Hamiltonian

$$H_0 = \frac{1}{2}(p^2 + x^2). \quad (55)$$

---

<sup>3</sup> The asymptotic behavior of  $W$  guarantees that the ground states of the two systems are in one to one correspondence and none has zero energy for  $a_1 = a_2 = 0$ .

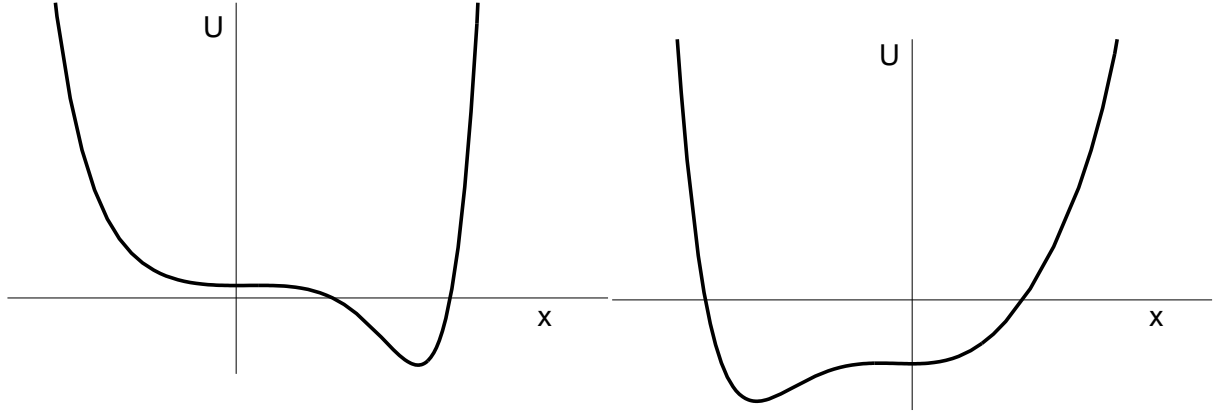


Fig. 7. Pair of isospectral potentials (53) and (54) which are not isoperiodic

We know that the eigenvalues and eigenstates of such operator are given by

$$H_0 \varphi_n(x) = E_n \varphi_n(x) , \quad (56)$$

with

$$E_n = \left( n + \frac{1}{2} \right) , \quad n = 0, 1, 2, \dots , \quad (57)$$

and

$$\varphi_n(x) = (\sqrt{\pi} 2^n n!)^{-1/2} \mathcal{H}_n(x) \exp\left(-\frac{x^2}{2}\right) , \quad (58)$$

where  $\mathcal{H}_n(x)$  denotes the Hermite polynomial.

Let us now consider the system with the same spectrum, except the lowest eigenvalue  $E_0 = 1/2$ :

$$E_n = \left( n + \frac{1}{2} \right) , \quad n = 1, 2, \dots . \quad (59)$$

For this we perform a similarity transformation which maps  $\varphi_0(x)$  into  $\tilde{\varphi}_0(x)$  by the formula <sup>4</sup>

$$\tilde{\varphi}_0(x) = \frac{\varphi_0(x)}{\Phi(x)} \quad (60)$$

<sup>4</sup> A similar transformation based on modding out by any eigenstate  $\varphi_n$  can be formally achieved but because of the existence of nodes in the wave function  $\varphi_n$  the induced potential is not defined on the whole real line  $\mathbb{R}$  [29]



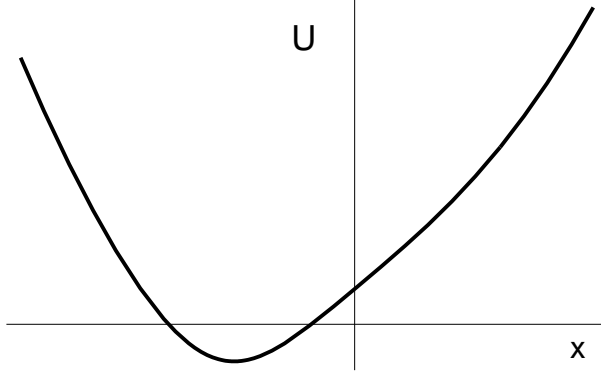


Fig. 8. Potential  $U(x) = \frac{1}{2}x^2 + 4\chi(x)(\chi(x) - x)$  with equally spaced spectrum where

$$\Phi(x) = \int_x^\infty \varphi_0^2(\xi) d\xi = \frac{1}{\sqrt{\pi}} \int_x^\infty e^{-\xi^2} d\xi, \quad (61)$$

for which

$$\Phi'(x) = -\varphi_0^2(x) = -\frac{1}{\sqrt{\pi}} e^{-x^2}. \quad (62)$$

Note that

$$\Phi(x) = \begin{cases} 1 & \text{at } x \rightarrow -\infty \\ \frac{1}{2x} \varphi_0^2(x) & \text{at } x \rightarrow \infty \end{cases} \quad (63)$$

and so,

$$\tilde{\varphi}_0(x) = \begin{cases} \varphi_0(x) & \text{at } x \rightarrow -\infty \\ \frac{2x}{\varphi_0(x)} & \text{at } x \rightarrow \infty \end{cases}, \quad (64)$$

and then  $\int_{-\infty}^\infty |\tilde{\varphi}_0(x)|^2 dx = \infty$ . Hence,

$$H_0 \tilde{\varphi}_0(x) = \frac{1}{2} \tilde{\varphi}_0(x), \quad \text{at } x \rightarrow \pm\infty. \quad (65)$$

The function  $\tilde{\varphi}_0(x)$  is the solution of the equation

$$H \tilde{\varphi}_0 = \frac{1}{2} \tilde{\varphi}_0, \quad (66)$$

where

$$H = \frac{1}{2} p^2 + U(x) \quad (67)$$

and

$$U(x) = U_0(x) + U_1(x) \quad (68)$$

with

$$U_0(x) = \frac{1}{2} x^2, \quad U_1(x) = -2 \frac{d^2}{dx^2} \log[\operatorname{erfc}(x)] = -4 \chi(x)(\chi(x) - x), \quad (69)$$

where the functions  $\operatorname{erfc}(x)$  and  $\chi(x)$  are

$$\operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-\xi^2) d\xi, \quad (70)$$

which satisfies

$$\operatorname{erfc}(x) = \begin{cases} 2 & x \rightarrow -\infty \\ 1 & x = 0 \\ \frac{1}{\sqrt{\pi} x} e^{-x^2} & x \rightarrow \infty \end{cases} \quad (71)$$

and

$$\chi(x) = (\sqrt{\pi} \operatorname{erfc}(x))^{-1} \exp(-x^2); \quad \chi(x) \approx x \quad \text{at } x \rightarrow \infty. \quad (72)$$

Notice that the new potential  $U(x) = U_0(x) + U_1(x)$  is neither shear equivalent to the harmonic oscillator  $\frac{1}{2}x^2$  nor isochronous.

It can be shown that  $H_0$  and  $H - \mathbb{1}$  have the same spectra. Indeed,

$$H \tilde{\varphi}_n(x) = E_n \tilde{\varphi}_n(x), \quad n = 1, 2, \dots \quad (73)$$

where

$$\tilde{\varphi}_n(x) = \varphi_n(x) - \sqrt{\frac{2}{n}} \chi(x) \varphi_{n-1}(x), \quad (74)$$

and the functions  $\tilde{\varphi}_n(x)$  are normalized:

$$\int_{-\infty}^{\infty} |\tilde{\varphi}_n(x)|^2 dx = 1 \quad (75)$$

and satisfy the completeness condition

$$\sum_{n=1}^{\infty} \tilde{\varphi}_n(x) \tilde{\varphi}_n(y) = \delta(x - y) . \quad (76)$$

i.e the Hamiltonians  $H_0$  and  $H - \mathbb{1}$  are isospectral. The peculiarity of this case is that the two isospectral potentials are neither classically isoperiodic nor related by a Darboux transformation.

Finally, it is also remarkable that the two families of isospectral rational potentials connected by Joukowski transformations as in Theorem 4 are in general not isospectral. Only the case of isochronous potential the half harmonic oscillator and the potentials (51) present the same quantum spectrum.

## Acknowledgments.

We thank F. Falceto, C. Farina, M. Rañada, A. Seguí and G. Sierra for discussions. Support of projects BFM-2003-02532, FPA-2003-02948, DGA2006 Grupo de Altas Energías and SAB2003-0256 is acknowledged.

## References

- [1] M. Asorey, A. Ibort and G. Marmo, “*Global Theory of Quantum Boundary Conditions and Topology Change*”, Int. J. Mod. Phys. **A 20** (2005) 1001–1025.
- [2] M. Asorey, A. Ibort and G. Marmo, “*Boundary Conditions and Path Integral*”, Proceedings of A. Galindo Festschrift, Eds. Alvarez-Estrada *et al*, Madrid (2004) 165–173.
- [3] N.H. Abel, “*Auflösung einer mechanischen Aufgabe*”, J. Reine Angew. Math. **1** (1826) 153–57.
- [4] R. Subramanian and K.V. Bhagwat, “*A lower bound for ground-state energy by Steiner symmetrisation of the potential*”, J. Phys. A: Math. Gen. **20**, 69–78 (1987)
- [5] S. Bolotin and R.S. MacKay, “*Isochronous potentials*”, in: *Localization and energy transfer in nonlinear systems*, p. 217–224, eds. L. Vázquez, R.S. MacKay and M.P Zorzano, World Sci. (2003).

- [6] F. Calogero, “*Two new classes of isochronous Hamiltonian systems*”, J. Nonlin. Math. Phys. **11** (2004) 208–222.
- [7] Ch. Huygens, “*Horologium Oscillatorium*”, Paris (1673)
- [8] O. A. Chalykh and A. P. Veselov, “*A remark on rational isochronous potentials*”, J. Nonlin. Math. Phys. **12** Suppl. 1 (2005) 179–183.
- [9] V.M. Eleonskii, V.G. Korolev and N.E. Kulagin, “*On a classical analog of the isospectral Schrödinger problem*”, JETP Lett. **65** (1997) 889–93.
- [10] J. Dorignac, “*On the quantum spectrum of isochronous potentials*”, J. Phys. A:Math. Gen. **38** (2005) 6183–210.
- [11] L.D. Landau and E.M. Lifshitz, “*Mechanics*”, Pergamon Press, London (1981).
- [12] B.F. Kimball, *Three theorems applicable to vibration theory*, Bull. Amer. Math. Soc. **38**, 718–23 (1933).
- [13] B.F. Kimball, *Note on a previous paper*, Bull. Amer. Math. Soc. **39**, 386 (1933)
- [14] A.H. Carter, “*A class of inverse problem in physics*”, Amer. J. Phys. **68** (2000) 698–703.
- [15] P. Appell, “*Traité de mécanique rationnelle*”, Vol **1**, Gauthiers-Villars, Paris (1902).
- [16] F.H. Stillinger and D.K. Stillinger, “*Pseudoharmonic oscillators and Inadequacy of Semiclassical Quantization*”, J. Phys. Chem. **93** (1989) 6890–92.
- [17] E.T. Ospanowski and M.G. Olsson, “*Isynchronous motion in classical mechanics*”, Amer. J. Phys. **55** (1987) 720–25.
- [18] P. Mohazzabi, “*On classical and quantum harmonic potentials*”, Can. J. Phys. **78** (10) (2000) 937–946.
- [19] G. Ghosh and R. W. Hasse, “*Inequivalence of the classes of quantum and classical harmonic potentials: Proof by example*”, Phys. Rev. **D 24** (1981) 1027–29.
- [20] M.M. Nieto and L.M. Simmons, “*Coherent states for general potentials. I. Formalism*”, Phys. Rev. **D 20** (1979) 1321–31
- [21] M.M. Nieto and L.M. Simmons, “*Coherent states for general potentials. II. Confining one-dimensional examples*”, Phys. Rev. **D 20**, 1332–41 (1979)
- [22] M.M. Nieto and V.P. Gutschick, “*Inequivalence of the classes of classical and quantum harmonic potentials: Proof by example*”, Phys. Rev. **D 23** (1981) 922–926.
- [23] R.W. Robinett, *Quantum Mechanics*, Oxford U.P., 1997.
- [24] G. Darboux, “*Sur une proposition relative aux équations linéaires*”, Comptes Rendues **94** (1882) 1456–1459.
- [25] P.B. Abraham and H.E. Moses, “*Changes in potentials due to changes in the point spectrum: Anharmonic oscillators with exact solutions*”, Phys. Rev. **A 22** (1980) 1333–40.

- [26] B. M. Levitan, “*Sturm-Liouville operators on the entire real axis with the same discrete spectrum*”, Math. USSR-Sb. **60** (1988) 77–106.
- [27] H.P. McKean and E. Trubowitz, “*The isospectral class of the quantum mechanical harmonic oscillator*”, Commun. Math. Phys. **82** (1981) 471–495.
- [28] A.M. Perelomov and Ya. B. Zel’dovich, “*Quantum Mechanics: Selected Topics*”, World Scientific (1998).
- [29] R. Jost and W. Kohn, “*Equivalent potentials*”, Phys. Rev. **88** (1952) 382–385.